COOLING OF A LAYER DURING STEFAN – BOLTZMANN RADIATION FROM ITS SURFACE

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A solution to the problem of the cooling of a uniformly heated spatial layer is given. It is assumed that the temperature inside the body follows the law of heat conduction while the surface radiates according to the Stefan–Boltzmann law.

We consider a homogeneous body the temperature $U(x, t)$ of which depends only on time t and one space coordinate x, the latter varying from 0 to $2l$.

This function satisfies the equation of heat conduction:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\cdot\frac{\partial u}{\partial t}, \quad a^2 = \frac{k}{c\rho} - \text{const}, \tag{1}$$

where k is the thermal conductivity; ρ is the density; and c is the specific heat of the given body. If $u(x, t)$ denotes the absolute temperature of the body and radiation is transmitted into vacuum, then the thermal flux at the body surface

$$k\frac{\partial u}{\partial x}\bigg|_{\substack{x=0\\x=2l}},$$

according to the Stefan–Boltzmann law, will be proportional to the fourth power of the temperature:

$$k\frac{\partial u}{\partial x}\bigg|_{x=0} = \sigma[u(0, t)]^4, \tag{2}$$

$$-k\frac{\partial u}{\partial x}\bigg|_{x=2l} = \sigma[u(2l, t)]^4. \tag{3}$$

The initial temperature distribution $u(x, 0)$ inside the body is equal to T_0:

$$u(x, t)|_{t=0} = T_0. \tag{4}$$

Since the initial values and the boundary conditions are symmetrical with respect to the point $x = l$,

$$\frac{\partial u}{\partial x}\bigg|_{x=l} = 0. \tag{5}$$

We denote the temperature gradient on the surface by $\nu(t)$:

$$\nu(t) = \frac{\partial u(x, t)}{\partial x}\bigg|_{x=0}.$$

We then must solve Eq. (1), the equation of heat conduction, with the boundary conditions

$$\frac{\partial u}{\partial x}\bigg|_{x=0} = \nu(t), \tag{6}$$

$$\frac{\partial u}{\partial x}\bigg|_{x=l} = 0 \tag{7}$$

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and the initial values (4). It now remains to determine $\nu(t)$ so that condition (2) be satisfied. For this purpose we reduce the heat-conduction equation (1) to a simpler parabolic equation by substituting $y = a^2 t$:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0.$$

Substituting $\eta = a^2\tau$ will reduce the conjugate equaton to

$$\frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial u}{\partial \eta} = 0.$$

Let us consider function

$$F(\zeta, \eta; x, y) = \begin{cases} \dfrac{1}{\sqrt{y-\eta}} \sum\limits_{n=-\infty}^{\infty} \left[\exp\left(-\dfrac{1}{4} \cdot \dfrac{(x-\zeta+2nl)^2}{y-\eta}\right) + \exp\left(-\dfrac{1}{4} \cdot \dfrac{(x+\zeta+2nl)^2}{y-\eta}\right)\right] & \text{for } y > \eta, \\ 0 & \text{for } y \leqslant \eta. \end{cases}$$

We now apply the Green formula for the heat-conduction operator to functions $u(\zeta, \eta)$ and $F(\zeta, \eta; x, y+h)$ $(h > 0)$, integrate over the contour ABQP (Fig. 1), move to the limit $h \to 0$, and use the Poisson formula to obtain

$$2\sqrt{\pi}u(x, y) = \int\limits_{AB} uF d\zeta + \int\limits_{PA} \left(F \frac{\partial u}{\partial \zeta} - u \frac{\partial F}{\partial \zeta}\right) d\eta + \int\limits_{BQ} \left(F \frac{\partial u}{\partial \zeta} - u \frac{\partial F}{\partial \zeta}\right) d\eta.$$

Considering the initial and the boundary conditions, while expanding function $F(\zeta, \eta; x, y)$ into a Fourier series:

$$F(\zeta, \eta; x, y) = \frac{2\sqrt{\pi}}{l} + \frac{4\sqrt{\pi}}{l} \sum_{n=1}^{\infty} \exp\left[-\frac{\pi^2 n^2}{l^2}(y-\eta)\right] \cos\frac{n\pi}{l} x \cdot \cos\frac{n\pi}{l} \zeta,$$

and transforming back to the original variables, we obtain the solution in the form:

$$u(x, t) = T_0 - \frac{a}{\sqrt{\pi}} \int\limits_0^t \frac{\nu(\tau)}{\sqrt{t-\tau}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{1}{4} \frac{(x+2nl)^2}{a^2(t-\tau)}\right] d\tau,$$

$$0 < x < l, \quad 0 \leqslant \tau \leqslant t,$$

which is convenient for calculations at small values of t. For calculations at large values of t we have the solution in the form

$$u(x, t) = T_0 - \frac{a^2}{l} \int\limits_0^t \nu(\tau) \left[1 + 2\sum_{n=1}^{\infty} \exp\left[-\frac{\pi^2 n^2 a^2}{l^2}(t-\tau)\right] \cos\frac{n\pi}{l} x d\tau.\right.$$

As has been mentioned already,

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \nu(t).$$

In order to satisfy condition (2), it is necessary that function $\nu(t)$ satisfy the relation

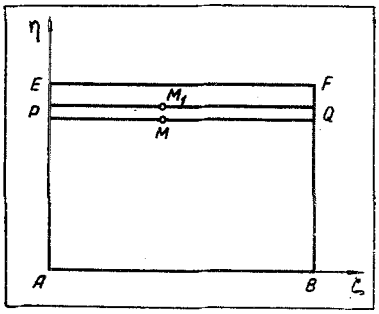
$$k\nu(t) = \sigma\left\{T_0 - \frac{a}{\sqrt{\pi}} \int\limits_0^t \frac{\nu(\tau)}{\sqrt{t-\tau}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{n^2 l^2}{a^2(t-\tau)}\right] d\tau\right\}^4, \tag{8}$$

which is a nonlinear integral equation in the function $\nu(t)$.

We take T_0 outside the brackets, divide by σT_0^4, and introduce new variables z, ζ letting

$$t = \gamma^2 z, \quad \tau = \gamma^2 \zeta,$$

and a new function



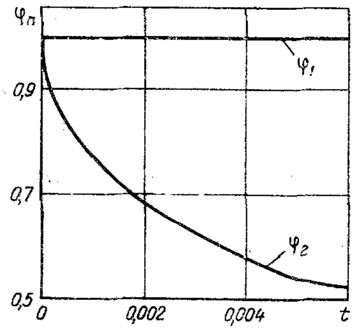


Fig.1. Contour of integration.

$$\varphi(z) = \nu(\gamma^2 z) \cdot \frac{k}{\sigma T_0^4},$$

where

$$\gamma = \sqrt{\pi}\,\frac{T_0}{a} \cdot \frac{k}{\sigma T_0^4},$$

so that

$$\varphi(z) = \left\{1 - \int_0^z \frac{\varphi(\zeta)}{\sqrt{z-\zeta}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{n^2 l^2}{a^2\gamma^2(z-\zeta)}\right] d\zeta\right\}^4. \tag{9}$$

If we find $\varphi(z)$, then

$$\nu(t) = \frac{\sigma T_0^4}{k}\,\varphi\left(\frac{t}{\gamma^2}\right).$$

The nonlinear integral equation (9) has a unique bounded solution [1, pp.461-469] on the interval $(0, z_0)$.

Another aspect of successive approximations is to be noted here. If we consider only those functions $\varphi(z)$ and those intervals $0 \le \zeta \le z$ for which

$$\int_0^z \frac{\varphi(\zeta)}{\sqrt{z-\zeta}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{n^2 l^2}{a^2\gamma^2(z-\zeta)}\right] d\zeta < 1,$$

then the successive approximations based on the relation

$$\varphi_n(z) = \left[1 - \int_0^z \frac{\varphi_{n-1}(\zeta)}{\sqrt{z-\zeta}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{n^2 l^2}{a^2\gamma^2(z-\zeta)}\right) d\zeta\right]^4$$

approach the solution to Eq. (9) from different directions.

Calculations yield the following result:

$$\varphi_0(z) = 0,$$

$$\varphi_1(z) = 1,$$

$$\varphi_2(z) = \left[1 - 2\sqrt{z} - 4\sum_{n=1}^{\infty} \int_0^{\sqrt{z}} \exp\left(-\frac{n^2 l^2}{a^2\gamma^2 t^2}\right) dt\right]^4.$$

For large values of z, the function $\varphi(z)$ is determined by the equation

$$\varphi(z) = \left[1 - \int_0^z \left(1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{\pi^2 n^2 a^2 \gamma^2}{l^2}(z-\zeta)\right)\right)\varphi(\zeta)\,d\zeta\right]^4.$$

The trend of these curves is shown in Fig.2, with

$$z = \frac{t}{\gamma^2} \text{ and } \gamma = \frac{\sqrt{\pi}}{a} T_0 \frac{k}{\sigma T_0^4}.$$

If $T_0 = 3000°K$, $k = 0.006$, $l = 2$ m, $a^2 = 0.007$ m^2/sec, and $\sigma = 1.2 \cdot 10^{-12}$ W/m$^2 \cdot$deg^4, then $\gamma = 3.5$ and one unit on the time axis is in our case equal to 12 sec.

The method of solution outlined is applicable also to the case where the body receives a constant supply of heat from an external source and its initial temperature remains constant.

NOTATION

$u(x, t)$	is the body temperature;
T_0	is the absolute temperature;
x	is the space coordinate;
t	is the time, sec;
$2l$	is the width of the spatial layer, m;
k	is the thermal conductivity, W/m $\cdot$ deg;
ρ	is the density, kg/m^3;
c	is the specific heat, J/kg $\cdot$ deg;
a	is the thermal diffusivity, m^2/sec;
σ	is the Stefan–Boltzmann constant, W/cm^2 $\cdot$ deg^4;
$\nu(t)$	is the temperature gradient at the body surface;
$\varphi(z), F(\xi, \eta; x, y)$	are functions;
ξ, η, z	are variables;
γ	is a constant;
h	in some number;
τ	is the variable of integration;
n	is a natural number.

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